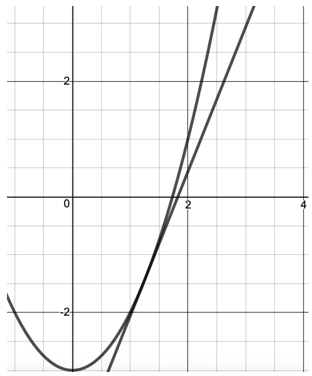
SECTION 4.8: NEWTON'S METHOD

RECALL: A **zero** of a function f is a solution to the equation $f(x) = 0$. That is, the zeros of a function correspond to the x-coordinates of the x-intercepts of the graph of $y = f(x)$.

BIG IDEA: Approximate the zeros of a function by finding the zeros of tangent lines.



RECALL: The equation of the tangent line at $x = a$ is: $y = f'(a)(x - a) + f(a)$.

Solving $y = f'(a)(x - a) + f(a) = 0$ gives: $x = a - \dfrac{f(a)}{f'(a)}$, provided $f'(a) \neq 0$.

QUESTION: What's happening on the graph of $y = f(a)$ at $(a, f(a))$ if $f'(a) = 0$?

NEWTON'S METHOD: Suppose we wish to approximate the zero of a differentiable function, f.

- Make an initial guess of the zero, x_0.
- Let $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$. That is, x_1 is the zero of the tangent line at $x = x_0$.
- Let $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$. That is, x_2 is the zero of the tangent line at $x = x_1$.

$\vdots$

- Let $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$. That is, x_{n+1} is the zero of the tangent line at $x = x_n$.

You can stop the process when the successive answers agree to as many digits as you need.

EXAMPLE 1: Let $f(x) = x^5 - x + 1$.

1. Find $f(-2)$ and $f(0)$ and use the IVT to prove f has a zero in the interval $(-2, 0)$.

   Since f is continuous on $[0, 2]$ and $f(-2) = -29 < 0 < 1 = f(0)$, the IVT gives $f(x) = 0$ in $(0, 2)$.

2. Use Newton's method to approximate the zero of f to 5 decimal places starting with $x_0 = -1$.

   HINT: On desmos define a function $z(x) = x - \dfrac{f(x)}{f'(x)}$ ...

   We get: $x_1 \approx -1.178459\ldots$, $x_2 \approx -1.167537\ldots$, $x_3 = -1.167304\ldots$, $x_4 = -1.167303\ldots$

   Hence, to 5 decimal places, the zero is $x = -1.16730$.

EXAMPLE 2: The monthly payment on a \$250,000 home for a 30-year fixed interest rate loan is:

$$p(i) = \frac{250000\,i}{1-(1+i)^{-360}}$$

where i is the monthly interest rate, obtained by dividing the annual interest rate, r by 12: $i = \dfrac{r}{12}$.

Use Newton's Method to determine i (and hence, r) if the monthly payment is to be no more than \$1500.

Ans: Apply Newton's Method to $f(i) = p(i) - 1500$ gives $i \approx 0.0050058$ and hence $r \approx 0.06006984$ so $r \approx 6\,\%$.

EXAMPLE 3: To find the square root of a positive number, p, the Babylonians used the following method:

- Make an initial guess x_0.
- Iterate according to the formula: $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{p}{x_n}\right)$.

1. Starting with initial guess $x_0 = 1.5$, approximate $\sqrt{2}$ by calculating x_3.

   Calculate the relative percentage error of using x_3 to approximate $\sqrt{2}$.

   $x_1 = 1.41\overline{6}$, $x_2 \approx 1.41421568627$, $x_3 \approx 1.41421356237$, $\dfrac{\left|x_3 - \sqrt{2}\right|}{\sqrt{2}} \times 100\% = 1.1276404038 \times 10^{-10}\,\%$

2. Derive the Babylonian Method using Newton's Method on the function $f(x) = x^2 - p$.

   $f'(x) = 2x$, so $L\left(x_n\right) = 2x_n\left(x - x_n\right) + x_n^2 - p$. Solving $L(x) = 0$ gives:

$$x_{n+1} = x_n + \frac{p - x_n^2}{2x_n} = x_n + \frac{p}{2x_n} - \frac{x_n^2}{2x_n} = x_n + \frac{p}{2x_n} - \frac{x_n}{2} = \frac{x_n}{2} + \frac{p}{2x_n} = \frac{1}{2}\left(x_n + \frac{p}{x_n}\right) \checkmark$$

EXAMPLE 4: Newton's Method won't work if $f'\left(x_n\right) = 0$. But it can also fail in other amusing ways.

Attempt to solve $x^3 + 2x = 4x^2 + 2$ using Newton's Method with an initial guess of $x_0 = 0$ and see what happens!

HOMEWORK: Section 4.8: 9 - 49 every other odd.